The prisoner's dilemma in structured scale-free networks

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Abstract
The conventional wisdom is that scale-free networks are prone to cooperation
spreading. In this paper we investigate the cooperative behavior on the
structured scale-free network. In contrast to the conventional wisdom that scale-
free networks are prone to cooperation spreading, the evolution of cooperation
is inhibited on the structured scale-free network when the prisoner's dilemma
(PD) game is modeled. First, we demonstrate that neither the scale-free
property nor the high clustering coefficient is responsible for the inhibition of
cooperation spreading on the structured scale-free network. Then we provide
one heuristic method to argue that the lack of age correlations and its associated
'large-world' behavior in the structured scale-free network inhibit the spread
of cooperation. These findings may help enlighten further studies on the
evolutionary dynamics of the PD game in scale-free networks.

PACS numbers: 87.23.Kg, 89.75.Hc, 87.23.Kg, 02.50.Le

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Cooperation is an essential factor for the evolution of species. For decades, scientists have
been dealing with the problem of understanding the emergence of cooperation [1–6]. In this
field of investigation, evolution game theory has become a powerful tool [7–9]. One of the
most frequently used metaphors is the prisoner's dilemma (PD) game [10] that is modeled to
simplify individual interactions when selfish actions provide a short-term higher payoff. In
each round two players are involved in the game. The two strategies for one player are to
become a cooperator or a defector. A cooperator is someone who pays some cost for another

individual to receive a benefit. A defector has no cost and does not deal out benefits. The cost and benefit are measured in terms of fitness. The combinations of the strategies will be of great difference. When both of the players choose to cooperate or to defect, each of the two will get benefit R or P, respectively. When one player chooses to cooperate and the other chooses to defect, the cooperator will get S, while the defector will get T. In the PD game, the order for the payoffs with different combinations of strategies is $T > R > P > S$. So the best strategy is to defect regardless the opponent's strategy and assuming that strategies are allowed to spread within the population according to their payoffs [7].

Recently, biological experiments have demonstrated that the evolution dynamics is related to the topological structure of the media, on the top of which the evolution dynamics are performed [11]. Earlier work [2, 4, 12, 13] mainly focused on the two-dimensional lattice, which is modeled for the homogeneous network. In these studies, a player is constrained to play solely with its nearest neighbors. All the studies reported that unlike in unstructured populations, cooperators and defectors can coexist in the lattice indefinitely. However, the regular network is not suitable for modeling real networks since a lot of empirical studies [14–17] have uncovered that the degree distribution $P(k)$ of many real-life systems complies with a power-law form $P(k) \sim k^{-\lambda}$ with $2 < \lambda \leqslant 3$. The scale-free property shows that everyone plays a different role in the network, which describes a kind of social diversity [18]. A famous model for generating the scale-free network is the Barabási–Albert (BA) model [19].

The evolution of cooperation on scale-free networks has been explored in [20–25], which exhibited that cooperation is the dominating trait throughout the entire range of parameters. In this paper, we focus on the evolution of cooperation on a structured scale-free network, which demonstrates negative age correlations [26]. First, we introduce the algorithm for constructing the network. Second, we simulate the PD game on the network, and compare the results with those of the BA network. In contrast to the results previously obtained for the BA network, we find that the structured scale-free network inhibits the spread of cooperation. Finally, we provide a heuristic argument to explain why the spread of cooperation is inhibited on the network considered, which is justified by extensive simulations.

2. Models

First let us introduce the celebrated BA model: initially there are m_0 connected nodes in the network. At each timestep, one node is added into the network and links with $m(m < m_0)$ existing nodes. The probability that an existing node i acquires a new link is $P(k_i) = \frac{k_i}{\sum_j k_j}$, where k_i is the degree of node i. Repeat this step until the desired network size is obtained. In the BA model, on the one hand, the chance that an old node receives a new link is proportional to its age; this phenomenon is called 'preferential attachment', also named 'age-correlations'. On the other hand, the long-range connections generated by the process decrease the distance between the vertices, leading to a small-world phenomenon: the average path length increases logarithmically with the network size [27].

The aging of nodes is particularly interesting. Recently, Klemm and Eguíluz found that for the scientific citation network [16], the age correlation is negative: the mean citation rate of papers decreases with the increase of their age [26], that is to say, old nodes have less probability to obtain links than those nodes just added into the network. To describe this phenomenon, they put forward a new model called the structured scale-free network (also called the highly clustered scale-free network). Their model is built as follows: initially there is a complete graph with m active nodes. At successive timesteps, one active node is added into

the network connected to all m active nodes, then one of $m+1$ active nodes is randomly chosen and deactivated with probability $P(k_j) = \frac{s}{a+k_j}$, where s is a normalization factor defined by $s^{-1} = \sum_i \frac{1}{a+k_i}$ with $a > 0$ being a constant, in which i belongs to the set of active nodes. Repeat the process until the network size reaches the required size.

According to the above process, we know the age correlations are greatly suppressed since in every round only the active nodes, not all existing nodes, have the chance to acquire new links. As a result, the linear topology of the structured scale-free network with local highly connected clusters forms a long chain [28]. The power-law exponent is $\lambda = 2 + \frac{a}{m}$. In this paper, we take $a = m$, and thus the structured scale-free network has the same degree distribution as the BA model with $\lambda = 3$ [19]. The clustering coefficient of the BA network approaches zero as the network size grows to be infinite, while in the structured scale-free network it is an asymptotic value $\frac{5}{6}$ for the case $a = m$.

After introducing the two models, we will study the evolutionary PD game on both models in the following section.

3. Simulations

In this section, we implement the finite population analog of replicator dynamics in the following PD simulations. Following [2], we make $T = b > 1$, $R = 1$ and $P = S = 0$ in the PD game, where b represents the temptation to defect, being typically constrained to the interval $1 < b \leqslant 2$. In each round of evolution, one node i plays the PD game with its directly connected neighbors, accumulating the payoff as P_i. Whenever the strategy of the player i with k_i neighbors is to be updated, a neighbor j is randomly selected from i's neighborhood. If $P_j > P_i$, the chosen neighbor j spreads its strategy to the player i with probability $\frac{P_j - P_i}{b(\max(k_i, k_j))}$, otherwise, the player i holds its strategy [20]. Simulations were carried out for a population with $N = 10\,000$ individuals. Initially, cooperator and defector strategies were distributed randomly among the players. Equilibrium frequencies of cooperators and defectors were obtained by averaging over 1000 generations after a transient time of 50 000 generations. The frequency of cooperators is a function of the parameter b for the PD game. Each data point corresponds to 100 simulations that are ten runs for ten different realizations of the same type of network specified by the appropriate parameters (the population size N and the average connectivity z).

Figure 1 shows the results for the PD game on both networks with different values of the average connectivity z. The BA scale-free network and the structured scale-free network are constructed by the above methods. The frequency of cooperators, f_c, is enhanced on the BA scale-free networks in the PD game, dominating over the entire range of b ($1 \leqslant b \leqslant 2$). Note that even when $2 < b \leqslant 3$, we have checked that the results will not change. When $1 < b \leqslant 1.4$, both of the networks are all-cooperator networks. As the temptation value b increases from 1.4, the frequency of cooperators in the structured scale-free network descends rapidly while the cooperative behavior is always flourishing in the BA network. Especially, for $b = 2$, when $z = 4$, f_c on the structured scale-free network is just half of that on the BA network; when $z = 8$, f_c decreases to 0 on the structured scale-free network, while the BA network still shows a remarkable survival of cooperation with $f_c \approx 0.4$. On the other hand, as z grows, the advantage of the scale-free network shrinks. Therefore, it can be observed from figure 1 that cooperators change into defectors more quickly with a smaller temptation value in the structured scale-free network, therefore, we can conclude that comparing with the BA network, the structured scale-free network decreases the frequency of cooperation, the reason for which will be analyzed in detail in the following section.
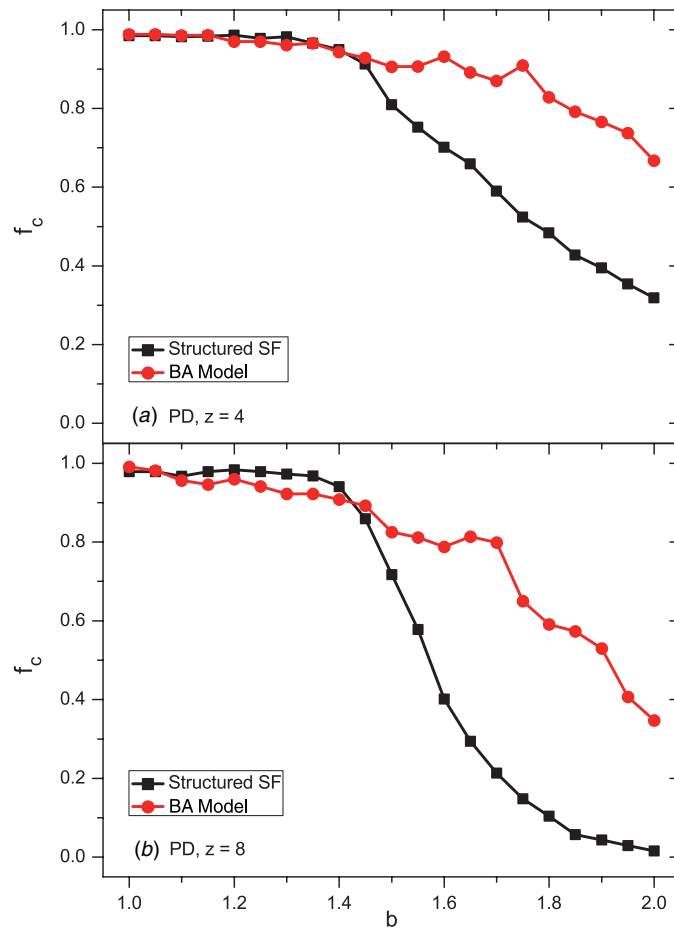
Figure 1. Frequency of cooperators f_c as a function of the advantage b of defectors for the PD game. The lines with filled squares show the results of the structured scale-free network in the PD game. The lines with dots are for the BA scale-free network. In the figure, z is the average connectivity of the network. According to the construction process of the structured scale-free network, we know $z = 2m$. The frequency of cooperation in the structured scale-free network is surpassed by that of the BA network in almost the entire range of b, especially when b is large enough.

4. Analysis

Why does the structured scale-free network inhibit the frequency of cooperators compared with the BA network? First, we are aware that both of the BA network and the structured scale-free network have the same power-law exponent ($\lambda = 3$) in the above simulations. So the scale-free property is a trivial factor that cannot affect the simulation results. Second, the clustering coefficient on the structured scale-free network is much higher than that in the BA network. Assenza *et al* in [29] find that the high clustering coefficient can help the enhancement of cooperation on the scale-free networks with a tunable value of the clustering coefficient [30], which is a different network model with the structured scale-free network. However, the high coefficient cannot help us to enhance the cooperation spreading on the structured scale-free network. So what is the root cause responsible for the inhibition of
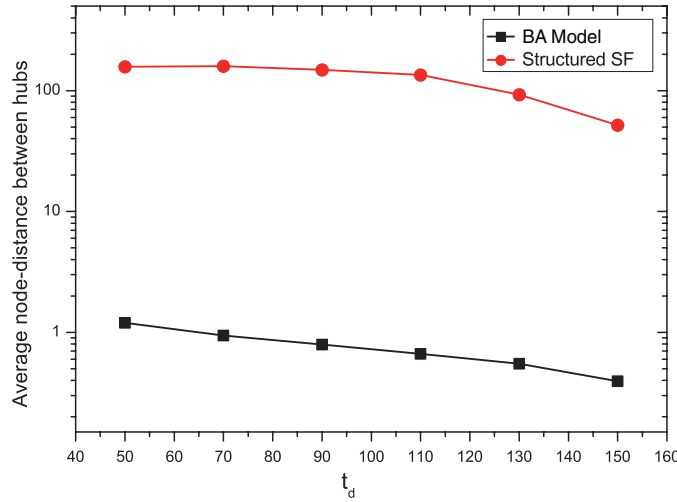
Figure 2. The average node distance between hubs as a function of the threshold of degree, t_d. A node i is called a hub node when its degree $k_i \geqslant t_d$. The solid line with dot symbols is for the structured scale-free network and the solid line with squares is for the BA network.

cooperation spreading on the structured scale-free network? From the models, we know the age correlations are greatly suppressed on the structured scale-free network compared with the BA network, which results in the fact that, in the BA network, a hub is usually linked to other highly connected nodes, while in the structured scale-free network, a hub is almost exclusively connected to low-degree nodes [31]. Also the diameter of the structured scale-free network increases linearly with the network size [28].

We can illustrate this phenomenon by calculating the node distance from one hub to others in the scale-free network as shown in figure 2. The value of the node distance from the origin to the destination is defined as the number of nodes on the shortest path between the two nodes. For example, the node distance of a pair of directly connected nodes is 0; if one of the shortest paths between nodes i and j bypasses another node k, then the node distance between i and j is 1, and so on. Figure 2 shows the average node distance between hubs as a function of threshold t_d. A node i can be qualified as a hub node only if its degree $k_i \geqslant t_d$. We can see that in the BA network, the hub nodes are usually interconnected with a shorter average node distance. While in the structured scale-free network, the average node distance between hubs is rather high. When $t_d = 50$, the node distance among hubs on the structured scale-free network is more than 100, while this value is just slightly more than one in the BA network. Hence, in the structured scale-free network, the hubs are connected through some low-degree nodes, which form an 'intermediate region'.

For the sake of simplicity, we model the two kinds of networks through two kinds of subgraphs shown in figure 3, where (*a*) shows a directly connected double star for the BA scale-free network and (*b*) is a double star with an intermediate region for the structured scale-free network. In figure 3(*a*), without losing generality, we assume that initially there are the unique defectors located on the right hub h_r and others (including the left hub h_l) are cooperators. Set $T = b$, $R = 1$ and $P = S = 0$. In the first generation of the networked PD game, the payoff of h_l is $\pi(h_l) = N$ and the payoff of h_r is $\pi(h_r) = Mb$. According to the evolutionary rule, the probability for the defector hub to invade the cooperator hub is

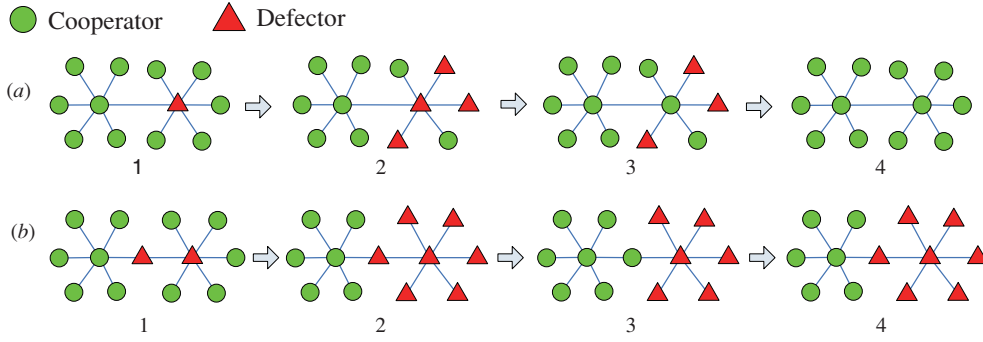$$P_1 = \frac{\pi(h_r) - \pi(h_l)}{b\max(M, N)} = \frac{Mb - N}{b\max(M, N)}. \tag{1}$$

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Figure 3. Evolution process on double-star graphs. (*a*) Directly connected double star for the BA network. The centers of the two stars are directly connected. The left hub h_l has $N-1$ leaves and the right one h_r has $M-1$ leaves. (*b*) Double star with an intermediate region for the structured scale-free network. Compared with (*a*), the hubs are not directly connected, but by the intermediate region. The degrees of hubs are the same as (*a*). In this figure, we just put one node on the intermediate region for the sake of simplifying the analysis.

Moreover, the payoff of some leaf on the right star l_r is $\pi(l_r) = 0$ and the defector hub can also invade its leaves. The probability for the defector hub to invade its leaf l_r is

$$P_2 = \frac{\pi(h_r) - \pi(l_r)}{b\max(1, M)} = 1. \tag{2}$$

Apparently, $P_1 < P_2$, hence, the leaves of the defector hub are easier to invade by the defection than the cooperative hub. After the defector hub has invaded some of its leaves and k ($k < M$) cooperator leaves survive, we get figure 3(*a*-2) where $\pi(h_r) = kb$. The probability that the defector hub invades the cooperator hub changes into

$$P_3 = \frac{\pi(h_r) - \pi(h_l)}{b\max(M, N)} = \frac{kb - N}{b\max(M, N)}. \tag{3}$$

Comparing with P_1, it can be found that the cooperator hubs are harder to invade by the defector hubs as k decreases. When $k = 0$, i.e., all cooperator leaves of the right star adopt the defector strategy, P_3 reaches its minimal value and the defector hub will definitely be invaded by the cooperator hub. Hence, after h_l changes into the cooperator, it will induce its neighbors learning its behavior and the cooperative strategy will spread to the whole double-star graph (figure 3(*a*-4)).

Through the structural analysis for the structured scale-free network, we point out that the intermediate region plays the crucial role for the evolution of cooperation. Hence, we add an intermediate node between the two hubs as shown in figure 3(*b*) to illustrate the effect of the intermediate region. We assume initially that the node x on the intermediate region is a defector and others are the same as in figure 3(*a*-1). In figure 3(*b*-1), $\pi(h_l) = N$, $\pi(h_r) = 0$ and $\pi(x) = b$. If x chooses h_l to update its strategy, the probability for x to turn into a cooperator is

$$P_4 = \frac{\pi(h_l) - \pi(x)}{b\max(2, N)} = \frac{N - b}{bN}. \tag{4}$$

Also according to the above analysis, the defector hub will invade all its leaves, and then we get figure 3(*b*-3). At this point, $\pi(x) = 1 < \pi(h_r) = b$. It is impossible for x to be invaded by the defector hub with probability

$$P_5 = \frac{\pi(h_r) - \pi(x)}{b\max(2, M)} = \frac{b - 1}{bM}. \tag{5}$$
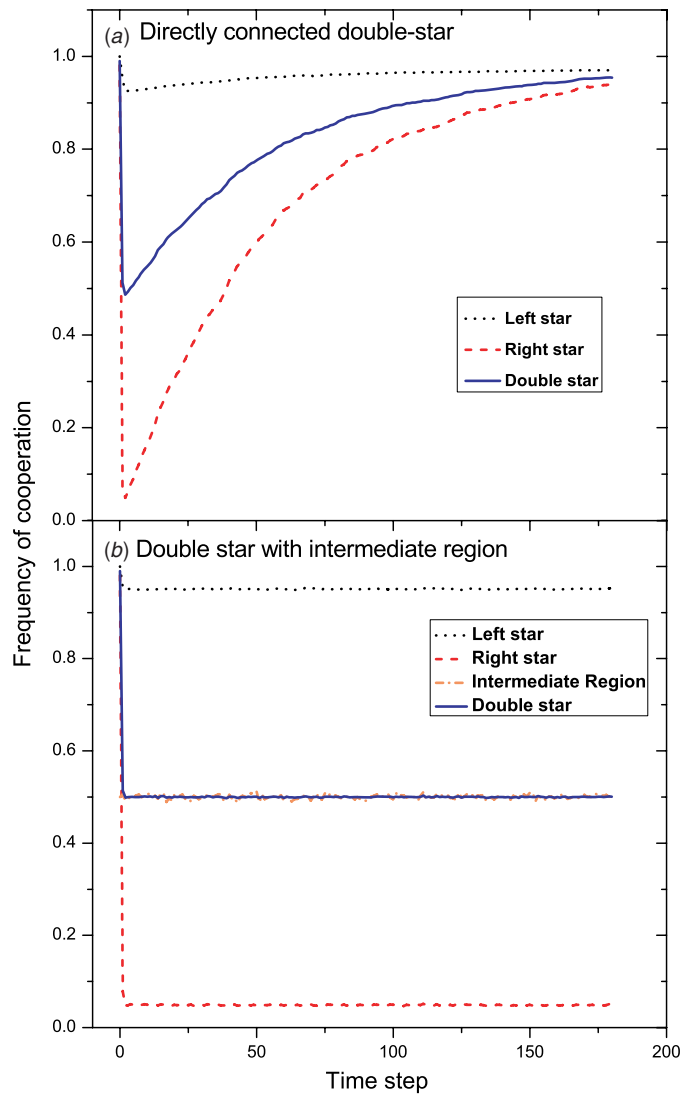
Figure 4. Evolution of cooperation under the PD game on the double-star graphs. Initially the cooperator hub h_l and the defector h_r are linked with the same number of leaves, $N = M = 50$. All leaves are cooperators. We fix $b = 1.5$. The simulations are averaged over 1000 runs. (*a*) The frequency of cooperation f_c, on the left star (dot line) stays at nearly 1.0. f_c in the right star (dashed line) first decreases to nearly 0 as defectors invade its leaves. Then at the lowest point, h_l spreads the cooperation strategy to h_r successfully. After that, the cooperators wipe up all the defectors on the right star. (*b*) We add a 10-node intermediate region between the double star as shown in the right corner. Initially the nodes on the intermediate region randomly choose their strategies. The left star contains all cooperators (dot line). f_c on the right star (dash line) decreases to nearly 0 (but not 0 because of the random effect) dramatically and stands still. f_c of the intermediate region (dash dot line) fluctuates around 0.5 as expected. The frequency of cooperation of the whole double star is shown as a solid line.

Therefore, the defector hub will receive the benefit from *x* when *x* adopts the cooperative strategy, furthermore, it will invade *x* and make *x* become defector again. Then the process drops into a loop shown between figures 3(*b*-3) and (*b*-4), alternatively. The nodes on

the intermediate region fluctuate their strategies and the defector hub is protected by the intermediate region against the invasion of cooperator hubs, resulting in the fact that the cooperative behavior is inhabited on the structured scale-free network for the PD game. In figure 3(*b*) we consider that the intermediate region only contains one node, and in the case of multiple nodes on the intermediate region the results do not change. Although we use the double-star graphs to illustrate the cooperation evolution on the scale-free networks, the analysis can heuristically reflect the microscopic organization of cooperation on the evolutionary dynamic of the PD game.

Furthermore, we have done extensive simulations shown in figure 4 to justify our above analysis. In the simulations we bring in a kind of random effect [22]. We assume that there are no direct connections among the leaves but they adopt a strategy from each other with probability 0.1, which is the random strategy adoption. From figure 4 it is observed that for the double-star graph, the defector hubs can be invaded by the cooperator hub and after which, its neighbors also become cooperators again. Whereas, for the double-star graph with the intermediate region, the leaves around the cooperative hub are cooperators and those around the defector hubs will hold onto the defection strategies. However, the intermediate region will change their strategies under the influence of their neighbors. Hence, the cooperators' frequency of the intermediate region is 0.5. Figure 4 confirms our conclusions discussed above, i.e., the isolation of communication among hubs in the structured scale-free network decreases the emergence of cooperation.

5. Conclusion

In this paper, we have investigated the evolution of cooperation on the structured scale-free network for the PD game. In contrast to the conventional wisdom that cooperation dominates on the BA scale-free network, the cooperative behavior is inhibited on the structured scale-free network. We find that neither the scale-free property nor the high clustering coefficient is the determining factor for the inhibition of cooperation spreading on the structured scale-free network. Then what is the root cause? Compared with the BA network, the age correlations are greatly suppressed on the structured scale-free network. As a result, hubs are usually linked to low-degree nodes and simulations have confirmed the result that the node distance among hubs is much larger on the structured scale-free network than on the BA network. Furthermore, to explain heuristically why the spread of cooperation is inhibited, we simplify the structured scale-free network as a double star connected by an intermediate region, which is formed by low-degree nodes. Through detailed analysis we showed that the lack of age correlations and associated 'large-world' behavior are responsible for the inhibition of cooperation on the structured scale-free network.

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